# CORRIGENDUM TO "EXISTENCE OF SOLUTIONS TO A DIFFUSIVE SHALLOW MEDIUM EQUATION"

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This short note explains how to fill a gap appearing in one of the arguments in our article "Existence of solutions to a diffusive shallow medium equation", see [1]. The need for a correction stems from the fact that in the proof of Lemma 5.2, one cannot directly use the monotonicity of the vector field $\tilde{A}_k$ to estimate the quantity $\langle \mathcal{A}w_m - \mathcal{A}w, w_m - w\rangle$ since in the monotonicity estimate (5.4), one needs to have the same first argument $v$ in $A_k(v,\xi)$ and $A_k(v,\eta)$. Therefore, to use monotonicity one needs to add and remove a term as we demonstrate below, leading to an extra term appearing in the estimates. We will show that the additional term vanishes in the limit $m \to \infty$, thus correcting the proof. We start by observing that the quantity $\langle \mathcal{A}w_m - \mathcal{A}w, w_m - w\rangle$ integrated over the time interval $[0,T]$ can be split into two terms as follows:

$$\begin{aligned}
\int_0^T \langle \mathcal{A}w_m - \mathcal{A}w, w_m - w\rangle \,\mathrm{d}t &= \iint_{\Omega_T} \big(\tilde{A}_k(w_m,\nabla w_m) - \tilde{A}_k(w,\nabla w)\big)\cdot(\nabla w_m - \nabla w)\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint_{\Omega_T} \big(\tilde{A}_k(w_m,\nabla w_m) - \tilde{A}_k(w_m,\nabla w)\big)\cdot(\nabla w_m - \nabla w)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{\Omega_T} \big(\tilde{A}_k(w_m,\nabla w) - \tilde{A}_k(w,\nabla w)\big)\cdot(\nabla w_m - \nabla w)\,\mathrm{d}x\,\mathrm{d}t \\
&=: I_1 + I_2.
\end{aligned}$$

For the integral $I_1$ we can now use the monotonicity property of $\tilde{A}_k$ to conclude that

$$I_1 \geq \begin{cases} c\displaystyle\iint_{\Omega_T} |\nabla w_m - \nabla w|^p \,\mathrm{d}x\,\mathrm{d}t, & p \geq 2 \\ c\displaystyle\iint_{\Omega_T\cap\{\nabla w \neq \nabla w_m\}} W_m^{p-2}|\nabla w_m - \nabla w|^2\,\mathrm{d}x\,\mathrm{d}t, & p < 2. \end{cases}$$

Thus, except for the remainder term $I_2$, we have the same estimate as in the original publication. We now show that $I_2$ vanishes in the limit $m \to \infty$ for a suitable subsequence $(w_{m_l})$ which means that the subsequent arguments made in the proof of Lemma 5.2 are valid. Since $(w_m)$ is a bounded sequence in $L^p(0,T;W^{1,p}(\Omega))$, we have using Hölder's inequality that

$$\begin{aligned}
|I_2| &= \Big| \iint_{\Omega_T} \big[T_k^\alpha(\tfrac{1}{k} + w_m) - T_k^\alpha(\tfrac{1}{k} + w)\big] |\nabla w + \nabla z|^{p-2}(\nabla w + \nabla z)\cdot(\nabla w_m - \nabla w)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \Big[\iint_{\Omega_T} \big|T_k^\alpha(\tfrac{1}{k} + w_m) - T_k^\alpha(\tfrac{1}{k} + w)\big|^{\frac{p}{p-1}} |\nabla w + \nabla z|^p \,\mathrm{d}x\,\mathrm{d}t\Big]^{\frac{p-1}{p}} \Big[\iint_{\Omega_T} |\nabla w_m - \nabla w|^p\Big]^{\frac{1}{p}}
\end{aligned}$$

*Date*: March 10, 2024.
2010 *Mathematics Subject Classification.* 35D30, 35K20, 35K65.
*Key words and phrases.* existence, shallow medium equation, doubly nonlinear parabolic equations.

$$\leq c\Big[\iint_{\Omega_T} \big|T_k^\alpha(\tfrac{1}{k}+w_m)-T_k^\alpha(\tfrac{1}{k}+w)\big|^{\frac{p}{p-1}}|\nabla w+\nabla z|^p\,\mathrm{d}x\,\mathrm{d}t\Big]^{\frac{p-1}{p}}.$$

Since the function $|\nabla w+\nabla z|^p$ is integrable and since $T_k^\alpha$ is bounded and continuous, in order to show that the last expression converges to zero it is sufficient by the dominated convergence theorem to show that $(w_m)$ (or rather a subsequence) converges pointwise a.e. to $w$. By a standard result in measure theory, it is sufficient to find a subsequence of $(w_m)$ which converges in $L^q(\Omega_T)$ for some exponent $q\geq 1$. We will find such a subsequence using the method of Lemma 3.5 in [2], which is based on a compactness result of Sobolev, see Theorem 1.4.3 in [3]. Examining the theorem, we see that since $\Omega_T$ is bounded and since $(w_m)$ is a bounded sequence in $L^2(\Omega_T)$, it only remains to show that

$$\lim_{|(y,h)|\to 0}\sup_{m\in\mathbb{N}}\iint_{\Omega_T}|w_m(x+y,t+h)-w_m(x,t)|^q\,\mathrm{d}x\,\mathrm{d}t=0, \tag{0.1}$$

for some exponent $q\geq 1$. Here $w_m$ is extended as zero outside $\Omega_T$. In order to do this, we investigate translations in space and time separately. For $0\leq\tau<\tau+h\leq T$ we have by (5.11) that

$$\begin{aligned}
&\|w_m(\tau+h)-w_m(\tau)\|^2_{L^2(\Omega)}\\
&=\int_0^h\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(\tau+t)-w_m(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&=2\int_0^h(w_m'(\tau+t),w_m(\tau+t)-w_m(\tau))\,\mathrm{d}t\\
&=2\int_0^h\langle F(\tau+t),w_m(\tau+t)-w_m(\tau)\rangle\,\mathrm{d}t-2\int_0^h\langle\mathcal{A}(w_m(\tau+t)),w_m(\tau+t)-w_m(\tau)\rangle\,\mathrm{d}t\\
&\leq 2\int_0^h\Big[\int_\Omega|f(x,\tau+t)|^{\frac{p}{p-1}}+|\nabla z(x)|^p\,\mathrm{d}x\Big]^{\frac{p-1}{p}}\|w_m(\tau+t)-w_m(\tau)\|_{W^{1,p}(\Omega)}\,\mathrm{d}t\\
&\quad+2\int_0^h\Big[\int_\Omega|\tilde{A}_k(w_m(\tau+t),\nabla w_m(\tau+t))|^{\frac{p}{p-1}}\,\mathrm{d}x\Big]^{\frac{p-1}{p}}\|w_m(\tau+t)-w_m(\tau)\|_{W^{1,p}(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Integrating this estimate over time and making use of Hölder's inequality we obtain

$$\begin{aligned}
&\int_0^{T-h}\|w_m(\tau+h)-w_m(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\
&\leq 2\int_0^h\int_0^{T-h}\Big[\int_\Omega|f(x,\tau+t)|^{\frac{p}{p-1}}+|\nabla z(x)|^p\,\mathrm{d}x\Big]^{\frac{p-1}{p}}\|w_m(\tau+t)-w_m(\tau)\|_{W^{1,p}(\Omega)}\,\mathrm{d}\tau\,\mathrm{d}t\\
&+2\int_0^h\int_0^{T-h}\Big[\int_\Omega|\tilde{A}_k(w_m(\tau+t),\nabla w_m(\tau+t))|^{\frac{p}{p-1}}\,\mathrm{d}x\Big]^{\frac{p-1}{p}}\|w_m(\tau+t)-w_m(\tau)\|_{W^{1,p}(\Omega)}\,\mathrm{d}\tau\,\mathrm{d}t\\
&\leq c\int_0^h\Big[\int_0^{T-h}\int_\Omega|f(x,\tau+t)|^{\frac{p}{p-1}}+|\nabla z(x)|^p\,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}}\Big[\int_0^T\|w_m(\tau)\|^p_{W^{1,p}(\Omega)}\,\mathrm{d}\tau\Big]^{\frac{1}{p}}\,\mathrm{d}t\\
&+c\int_0^h\Big[\int_0^{T-h}\int_\Omega|\tilde{A}_k(w_m(\tau+t),\nabla w_m(\tau+t))|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}}\Big[\int_0^T\|w_m(\tau)\|^p_{W^{1,p}(\Omega)}\,\mathrm{d}\tau\Big]^{\frac{1}{p}}\,\mathrm{d}t\\
&\leq ch\Big[\iint_{\Omega_T}|f|^{\frac{p}{p-1}}+|\nabla z|^p\,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}}\|w_m\|_{L^p(0,T;W^{1,p}(\Omega))}\\
&\quad+ch\Big[\iint_{\Omega_T}|\tilde{A}_k(w_m,\nabla w_m)|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}}\|w_m\|_{L^p(0,T;W^{1,p}(\Omega))}.
\end{aligned}$$

Using the definition of $\tilde{A}_k$ we can estimate the last expression upwards to obtain

$$\begin{aligned}\int_0^{T-h} \|w_m(\tau+h)-w_m(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \le ch\Big[\iint_{\Omega_T} |f|^{\frac{p}{p-1}} + |\nabla z|^p \,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}} \|w_m\|_{L^p(0,T;W^{1,p}(\Omega))} \\ + ch\Big[\iint_{\Omega_T} |\nabla w_m|^p + |\nabla z|^p \,\mathrm{d}x\,\mathrm{d}\tau\Big]^{\frac{p-1}{p}} \|w_m\|_{L^p(0,T;W^{1,p}(\Omega))}.\end{aligned}$$

Using the boundedness of the sequence $(w_m)$ in $L^p(0,T;W^{1,p}(\Omega))$ as well as the integrability properties of $f$ and $\nabla z$ we end up with

$$\int_0^{T-h} \|w_m(\tau+h)-w_m(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \le Ch.$$

Recalling that $w_m$ is extended as zero outside of $\Omega_T$ and that $\|w_m(t)\|_{L^2(\Omega)}$ is bounded uniformly in $t$, we also obtain

$$\iint_{\Omega_T} |w_m(x,t+h)-w_m(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \le Ch, \tag{0.2}$$

which is the desired estimate for the translation in time.

For translations in space, first note that we may choose the basis functions $v_j$ in $C_0^\infty(\Omega)$ for convenience. Then also $w_m$ will be smooth in the $x$-variable and $w_m$ will vanish outside $\Omega$. Consider first only a translation in the $j$th spatial coordinate direction. Let $x\in\Omega$. Then

$$\begin{aligned}|w_m(x+he_j,t)-w_m(x,t)| &\le \int_0^h |\partial_j w_m(x+se_j,t)|\,\mathrm{d}s \\ &\le h^{\frac{p-1}{p}}\Big[\int_0^h |\partial_j w_m(x+se_j,t)|^p\,\mathrm{d}s\Big]^{\frac{1}{p}} \\ &\le h^{\frac{p-1}{p}}\Big[\int_{\{s\in\mathbb{R}\,|\,x+se_j\in\Omega\}} |\partial_j w_m(x+se_j,t)|^p\,\mathrm{d}s\Big]^{\frac{1}{p}}.\end{aligned} \tag{0.3}$$

Integrating over $x$ and $t$ and utilizing the fact that $\Omega$ is bounded, we thus have

$$\iint_{\Omega_T} |w_m(x+he_j,t)-w_m(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \le Ch^{p-1}\iint_{\Omega_T}|\partial_j w_m|^p\,\mathrm{d}x\,\mathrm{d}t \le Ch^{p-1}, \tag{0.4}$$

where the last estimate follows from the fact that $(w_m)$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$. Combining (0.4) and (0.2) we can verify (0.1) for $q=\min\{2,p\}$, which completes the argument.

## References


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